TTIC 31260 - Algorithmic Game Theory (Spring 2024)

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Fair Division II: Indivisible Goods

1 Overview

In the last class we talked about fair division of divisible goods (cake cutting). Now we will look at partitioning indivisible goods. For example, dividing up an inheritance of m items among n people. In this case, we can't necessarily achieve "fairness" as defined in the last class, or envy-freeness, because there might be just one or two high-value items and we can't split them up. People will envy whoever gets those items. So, we will instead look at some relaxed goals. Much of this discussion is taken from Ruta Mehta's class notes at UIUC.

2 Setup and Definitions

We assume n players and m items. Each player i is assumed to have some monotonic valuation function over subsets: that is, if $S \subseteq T$ then $v_i(S) \leq v_i(T)$. Sometimes we will also assume valuation functions are additive. We will focus primarily on the algorithmic problem: given the valuation functions, how can we allocate the items to satisfy some desirable condition, but some of our algorithms will indeed be protocols as defined in the last class. We won't require each agent's value on the "grand bundle" of all items to equal 1 since the scaling won't matter for the questions we will care about. We will use $S_1, ..., S_n$ to denote an allocation with agent i getting S_i.

As noted above, we can't hope to achieve envy-freeness in this setting. Instead, here are two relaxed conditions we will consider.

Definition 1 (Envy-Free up to One Item (EF1)) *An allocation $S_1, ..., S_n$ satisfies EF1 if for all i, for all j, either S_j is empty or there exists some good $g \in S_j$ such that $v_i(S_i) \geq v_i(S_j \setminus \{g\})$.*

In other words, agent i might envy the bundle S_j received by agent j, but there exists at least one good that can be removed from it such that agent i would no longer envy it. Note that this removal is just a thought experiment in the definition of EF1 — we're not actually removing the item.

Here is another, stricter definition.

Definition 2 (Envy-Free up to Any Item (EFX)) *An allocation $S_1, ..., S_n$ satisfies EFX if for all i, j, we have $v_i(S_i) \geq v_i(S_j \setminus \{g\})$ for all $g \in S_j$.*

An interesting open question is whether, for additive valuations, an EFX allocation always exists. It's known to always exist for $n \leq 3$ but not known more generally. It's a puzzling question because it's just a property of a matrix V where $V_{ik} = v_i(k)$. Let's start by discussing EF1.

3 EF1 for additive valuations

As we briefly mentioned last time, there is a simple protocol for finding an EF1 allocation for the case of additive valuations.

Round Robin: players go in order, taking their favorite item from among those remaining.

Let's define s_{ik} to be the kth item taken by player i.

This is EF1 for players with additive valuations because for $i < j$, player i doesn't envy player j at all ($v_i(s_{i1}) \geq v_i(s_{j1}), v_i(s_{i2}) \geq v_i(s_{j2}), ...$), and for $i > j$, player i likes their own kth item at least as much as player j's $(k+1)$st item.

How might this break down for players with non-additive valuations? One problem is it's too myopic. For example, let's say there are two players with the same valuation function v, and for a nonempty set S of items, $v(S) = |S|$ if $|S| \leq 2$ or S contains item 2 or S contains item 4, else $v(S) = 2$. Then we could have player 1 takes item 1, player 2 takes item 2, player 1 takes item 3, player 2 takes item 4, player 1 takes item 5, player 2 takes item 6, player 1 takes item 7, player 2 takes item 8.

Next we'll look at a procedure that's EF1 for general monotone valuations.

4 EF1 for general monotone valuations [LMMS04]

To achieve an EF1 allocation for players with general monotone valuation functions we will allocate the items one at a time based on the "envy structure" of the allocation so far, and then occasionally switch bundles among the players. Unlike the case of additive valuations, this procedure is no longer a "protocol" for players to follow.

To discuss the algorithm, let's make a few definitions.

Definition 3 (partial allocation) *A partial allocation $S_1, ..., S_n$ is an allocation of some (but perhaps not all) of the m items.*

Definition 4 (envy graph) *Given a partial allocation $S_1, ..., S_n$, the envy graph has vertices $1, ..., n$ and a directed edge from i to j if i envies j. That is, the edge set is $\{(i, j) : v_i(S_j) > v_i(S_i)\}$.*

Suppose we inductively have a partial allocation satisfying EF1 (base case: the empty allocation satisfies EF1). The first key idea of the algorithm is that it is always safe to allocate the next item to any vertex of in-degree 0 (any "source") in this directed graph. That is because nobody envies a source (by definition of the graph), so no matter what item we give them, it will not create any violations of the EF1 condition.

So, as long as the graph has at least one source, we can continue to allocate items. But what if we get to a state where there are no longer any sources? Notice that this means the graph must have at least one cycle in it. This gets us to the second key idea.

Second key idea: given a cycle $i_1 \to i_2 \to ... \to i_k \to i_1$ in the envy graph, rotate the bundles: give S_{i_2} to player i_1, give S_{i_3} to player i_2, ..., give S_{i_1} to player i_k. Notice that each player on the cycle *is strictly better off than they were before.* Now, it could be that the cycle still exists, but if so, we just rotate the bundles again; we can keep doing it until the cycle breaks (which it has to at some point before everyone gets their original bundle back).

We can use this procedure to remove all the cycles in the envy graph. There are two ways to see this. One is that each rotation strictly improves the valuation of all players on the cycle and doesn't affect anyone else's valuation, so this can only happen a limited number of times (at most $O(n^2)$ times since there are only n different bundles in the graph – we aren't creating new bundles – so any given player can become happier at most $n - 1$ times). A second way to see this is that each time we break a cycle, the total number of edges in the envy graph drops by at least 1. That's because we are only rotating bundles around, so for any vertex not on the cycle, their out-degree hasn't changed (they still envy the same set of bundles — the only thing that changed is who owns those bundles). And for any vertex on the cycle, their out-degree can only drop (since they are getting happier).

Finally, notice that if the partial allocation satisfies EF1, then it still satisfies it after a rotation, since again the players on the cycle are strictly happier and they don't envy their old bundles.

So, that's the algorithm. We add items one at a time to any source in the graph, and if there is no source then we do rotations until we remove cycles and get a source again.

5 EFX

We can get an EFX allocation for two players using cut-and-choose. Specifically, player 1 partitions the items into two sets A and B such that player 1 would not have EFX-envy regardless of which set they get, and then player 2 chooses their favorite.

How can player 1 do the partition? If valuations are additive, then this is particularly easy. Start with A and B as empty sets, and sort the items from most valuable to least valuable according to player 1. Then, have player 1 go through the items one at a time in that order, adding each one into whichever of A or B is currently the least valuable. This maintains inductively that player 1 has no EFX-envy between the two sets.

What about for general monotone valuations? Here is one way. Start with some arbitrary partition A, B. Without loss of generality, say that $v_1(A) \leq v_1(B)$. Now, if there is some good $g \in B$ such that $v_1(A) < v_1(B \setminus \{g\})$, then move g into A. Notice that this either (1) strictly increases the minimum of player 1's valuations (if $v_1(A \cup \{g\}) > v_i(A)$) or (2) it keeps the minimum valuation the same but strictly reduces the *size* of the higher-valuation set. So, if we continue to do this (renaming the sets to keep A the set of minimum value) so long as such a good exists, this has to eventually halt.

For *additive* valuations, it is known how to achieve EFX for 3 players. It's not known if an EFX partition always exists for $n > 3$.

6 References

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